Dynamics of metronomes in a row on a moving plate

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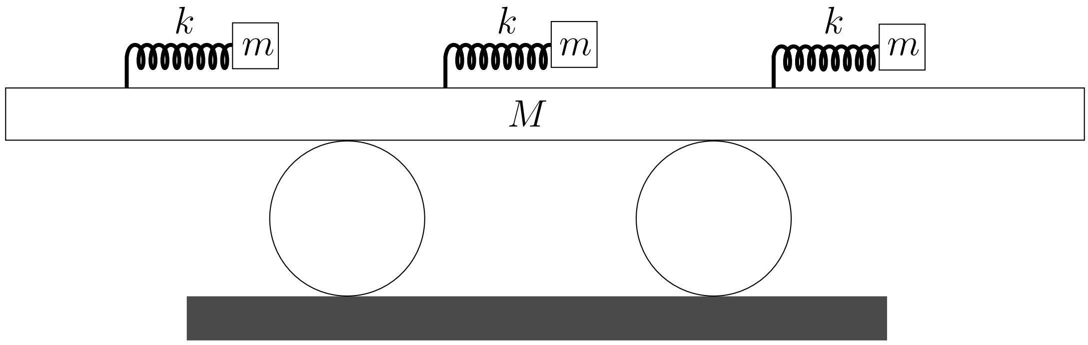
1 The problem

We consider a row of N harmonic oscillators attached to a freely moving plate of mass M. All the oscillators are identical with mass m and rigidity k. We denote $\omega_0 = \sqrt{k/m}$ their proper frequency.



2 Dynamics of the system

2.1 Conservative case

If x_j is the departure of the j^{th} oscillators from its equilibrium position with respect to the plate, and x is the position of the plate, then the kinetic energy of the system reads

$$\mathcal{T} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\sum_{j=1}^{N}\left(\dot{x}+\dot{x}_j\right)^2, \tag{1}$$

while the potential energy is

$$\mathcal{U} = \frac{1}{2}k\sum_{j=1}^{N}x_j^2. \tag{2}$$

We can write the Lagrangian of the system as $\tilde{\mathcal{L}} = \left(\mathcal{T} - \mathcal{U}\right)/m$ to end up with

$$\tilde{\mathcal{L}} = \frac{1}{2}\frac{M}{m}\dot{x}^2 + \frac{1}{2}\sum_{j=1}^{N}\left(\dot{x}+\dot{x}_j\right)^2 - \frac{1}{2}\omega_0^2\sum_{j=1}^{N}x_j^2. \tag{3}$$

We introduce the dimensionless time $t^\star = \omega_0 t$ and for the rest of Sect. 2.1, the upper dot denotes $d/dt^\star$. Redefining the Lagrangian as $\mathcal{L} = \tilde{\mathcal{L}}/\omega_0^2$, we have the final expression

$$\mathcal{L} = \frac{1}{2}\frac{M}{m}\dot{x}^2 + \frac{1}{2}\sum_{j=1}^{N}\left(\dot{x}+\dot{x}_j\right)^2 - \frac{1}{2}\sum_{j=1}^{N}x_j^2. \tag{4}$$

The momenta associated with the coordinates $x, x_1, \cdots, x_N$ are given by

$$\begin{aligned} X &= \frac{\partial L}{\partial \dot{x}} = \frac{M}{m}\dot{x} + \sum_{i=1}^{N}\left(\dot{x}+\dot{x}_i\right) \text{ and} \\ X_j &= \frac{\partial L}{\partial \dot{x}_j} = \dot{x}+\dot{x}_j. \end{aligned} \tag{5}$$

The Hamiltonian of the system is defined by $\mathcal{H} = (X, X_1, \cdots, X_n)\ {}^t(\dot{x}, \dot{x}_1, \cdots, \dot{x}_n) - \mathcal{L}$. It does not depend on x, which means that X is a first integral, corresponding to the total angular momentum of the system. It can be assumed without loss of generality that X is zero, and in that case, the Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\frac{m}{M}\left(\sum_{j=1}^{N} X_j\right)^2 + \frac{1}{2}\sum_{j=1}^{N}\left(X_j^2 + x_j^2\right). \tag{6}$$

The second term corresponds to a sum of N uncoupled oscillators while the first term contains the coupling. Intuitively enough, if M is much larger than m, then the coupling term vanishes and the oscillators do not interact with each other.

The equations of motions are given by

$$\dot{x}_j = \frac{\partial \mathcal{H}}{\partial X_j} \quad \text{and} \quad \dot{X}_j = -\frac{\partial \mathcal{H}}{\partial x_j} \tag{7}$$

and are linear. Writing $\mathfrak{X} = {}^t(X_1, \cdots, X_n, x_1, \cdots, x_n)$, we obtain

$$\frac{d\mathfrak{X}}{dt} = \omega_0 \begin{pmatrix} 0_N & -\mathbb{I}_N \\ \mathbb{I}_N + \dfrac{m}{M}\mathbb{1}_N & 0_N \end{pmatrix} \mathfrak{X} = \mathcal{M}\mathfrak{X}, \tag{8}$$

where $\mathbb{1}_N$ is the $N \times N$ matrix full of 1. The characteristic polynomial of $\mathcal{M}$ reads

$$\det\left(\lambda\mathbb{I} - \mathcal{M}\right) = \left(\lambda^2 + \omega_0^2\right)^{N-1}\left(\lambda^2 + \Omega_0^2\right), \qquad \text{where } \Omega_0^2 = \frac{k}{m} + N\frac{k}{M}, \tag{9}$$

and the eigenvalues of the system are $\pm i\omega_0$ of multiplicity $N-1$ and $\pm i\Omega_0$ of multiplicity 1. Without dissipation, the oscillators all have a quasiperiodic motion with frequencies ω_0 and Ω_0.

2.2 Dissipative case

Dissipation can be taken into account in the model by modifying the Euler-Lagrange equation. We write $\boldsymbol{q} = {}^t(x, x_1, \cdots, x_n)$ and we introduce a dissipation function $\tilde{\mathcal{D}}$ such that the work of the dissipative forces takes the expression

$$W = -\frac{\partial \tilde{\mathcal{D}}}{\partial \dot{\boldsymbol{q}}} \cdot \dot{\boldsymbol{q}}. \tag{10}$$

A simple model that disregards air drag and assumes that the dissipation forces are proportionnal to the speed yields to the introduction of two timescales τ_1 and τ_2 such that

$$\tilde{\mathcal{D}} = \frac{1}{2}\frac{M}{\tau_1}\dot{x}^2 + \frac{1}{2}\frac{m}{\tau_2}\sum_{j=1}^{N}\dot{x}_j^2. \tag{11}$$

The equations of motion with dissipation are given by

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\boldsymbol{q}}} = \frac{\partial \tilde{\mathcal{D}}}{\partial \dot{\boldsymbol{q}}}, \tag{12}$$

where $\tilde{\mathcal{L}} = \mathcal{T} - \mathcal{U}$ and $\mathcal{T}$ and $\mathcal{U}$ are given by Eqs. (1) & (2). We again introduce the dimensionless time $t^\star = \omega_0 t$ and redefine the upper dot as $d/dt^\star$. We redefine the Lagrangian and dissipative function by $\mathcal{L} = \tilde{\mathcal{L}}/\left(m\omega_0^2\right)$ and $\mathcal{D} = \tilde{\mathcal{D}}/\left(m\omega_0^2\right)$. The Lagrangian then takes the expression given by Eq. (4) and

$$\mathcal{D} = \frac{1}{2\tau_1}\frac{M}{m}\dot{x}^2 + \frac{1}{2\tau_2}\sum_{j=1}^{N}\dot{x}_j^2. \tag{13}$$