Mean motion resonances in planetary systems

Dynamics tutorial for master students at Geneva Observatory

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Contents

Abstract

Mean motion resonances (MMR) occur in planetary systems when the period ratio P_j/P_i between two planets is close to a rational number p/q. When this happens, linear combinations of angles of the form $p\lambda_i - q\lambda_j$, where λ_i and λ_j are related to the angular position of the planets, evolve slowly. This slow evolution allow the small planet−planet mutual interactions to build up over time, affecting the orbits over secular timescales.

One notable consequence of planetary MMR is the increase in the eccentricities of the orbits. In this tutorial, we develop a simple model of first-order MMR, showing the eccentricity increase and calculating the associated secular timescales.

1 Reduction to two degrees of freedom

We consider two planets of masses m_1 and m_2 orbiting a star of mass m_0. In this tutorial, $\iota = \sqrt{-1}$, vectors are in bold font, $\bar{\cdot}$ denotes the complex conjugated and ${}^t\cdot$ is the transpose operator. With $\boldsymbol{r}_j$ and $\boldsymbol{v}_j$ the heliocentric position and barycentric speed of planet j, respectively, the Hamiltonian of the problem can be written (Laskar and Robutel, 1995, Eq. (2.53))

$$H = \sum_{j=1}^{2} \left(\frac{\tilde{r}_j^2}{2\beta_j} - \frac{\mathcal{G} m_0 m_j}{r_j} \right) + \frac{\tilde{\boldsymbol{r}}_1 \cdot \tilde{\boldsymbol{r}}_2}{m_0} - \frac{\mathcal{G} m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}, \tag{1}$$

where the momenta $\tilde{\boldsymbol{r}}_j = \beta_j \boldsymbol{v}_j$ are conjugated to $\boldsymbol{r}_j$ and $\beta_j = m_0 m_j / (m_0 + m_j)$.

Q0 - *In the coplanar case ($\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are two-dimensional vectors), how many degrees of freedom does the Hamiltonian have?*

In the coplanar case, we introduce the Poincaré polar coordinates $(\Lambda_j, P_j; \lambda_j, p_j)$, where λ_j is the mean longitude, $p_j = -\varpi_j$ is the opposite of the longitude of the periapsis, $\Lambda_j = \beta_j \sqrt{\mu_j a_j}$ is conjugated to λ_j and $P_j = \Lambda_j \left(1 - \sqrt{1 - e_j^2}\right)$ is conjugated to p_j. a_j and e_j are the semi-major axis and eccentricity, while $\mu_j = \mathcal{G}(m_0 + m_j)$. In these coordinates, the Keplerian part of the Hamiltonian, due to star-planet interactions, reads

$$H_K(\Lambda_j) = \sum_{j=1}^{2} \left(\frac{\tilde{r}_j^2}{2\beta_j} - \frac{\mathcal{G} m_0 m_j}{r_j} \right) = -\sum_{j=1}^{2} \frac{\beta_j^3 \mu_j^2}{2\Lambda_j^2}. \tag{2}$$

We assume that the Poincaré polar coordinates $(\Lambda_j, P_j; \lambda_j, p_j)$ are canonical (see Laskar, 2017, for a demonstration). We introduce the Poincaré complex variables with momenta

$$x_j = \sqrt{P_j} e^{\iota \varpi_j} = \sqrt{P_j} e^{-\iota p_j}, \tag{3}$$

and generalized coordinates $\tilde{x}_j = -\iota \bar{x}_j$. These variables are canonical and proportional to the eccentricity, since

$$x_j = \sqrt{\frac{\Lambda_j}{2}} e_j e^{\iota \varpi_j} + \mathcal{O}(e_j^3). \tag{4}$$

In the coplanar case, the perturbative part of the Hamiltonian can be expanded in series of x_j and $\bar{x}_j$ as (Laskar and Robutel, 1995, Eq. 73)

$$H_P = \frac{\tilde{\boldsymbol{r}}_1 \cdot \tilde{\boldsymbol{r}}_2}{m_0} - \frac{\mathcal{G} m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \left(\sum_{\boldsymbol{u} \in \mathbb{N}^4} \mathcal{C}_{\boldsymbol{k},\boldsymbol{u}}(\Lambda_j) x_1^{u_1} x_2^{u_2} \bar{x}_1^{\tilde{u}_1} \bar{x}_2^{\tilde{u}_2} \right) e^{\iota(k_1\lambda_1 + k_2\lambda_2)}, \tag{5}$$

where $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$ and $\boldsymbol{u} = (u_1, u_2, \tilde{u}_1, \tilde{u}_2) \in \mathbb{N}^4$. Since the Hamiltonian governs the equations of motions, it should remain unchanged by a redefinition of the axes of the problem, that is to say, by a translation ϑ on the angles of the problem.

Q1 - *Inject the transformation $\lambda_j \leftarrow \lambda_j + \vartheta$ and $\varpi_j \leftarrow \varpi_j + \vartheta$ into Eq. (5) and show that H_P is invariant by the transformation only if $\mathcal{C}_{\boldsymbol{k},\boldsymbol{u}} = 0$ when $k_1 + k_2 + u_1 + u_2 - \tilde{u}_1 - \tilde{u}_2 \neq 0$.*

We obtain the so-called d'Alembert rule

$$\mathcal{C}_{\boldsymbol{k},\boldsymbol{u}} \neq 0 \quad \Rightarrow \quad k_1 + k_2 + u_1 + u_2 - \tilde{u}_1 - \tilde{u}_2 = 0. \tag{6}$$

In addition to the d'Alembert rule, the invariance of the Hamiltonian by the transformation $\lambda_j \leftarrow -\lambda_j$ and $\varpi_j \leftarrow -\varpi_j$ implies that

$$\mathcal{C}_{k_1,k_2,u_1,u_2,\tilde{u}_1,\tilde{u}_2} = \mathcal{C}_{-k_1,-k_2,\tilde{u}_1,\tilde{u}_2,u_1,u_2}, \tag{7}$$

which ensures that the Hamiltonian is real and contains only cosine terms. In Eq. (5), we truncate the summation to some order d in eccentricity. That is, any term such that $u_1 + u_2 + \tilde{u}_1 + \tilde{u}_2 > d$ is discarded. Furthermore, any term such that $k_1\lambda_1 + k_2\lambda_2$ is a fast circulating angle is discarded as well because the corresponding cosine term averages to zero over large timescales. In the solar system, Jupiter and Saturn are fairly close to resonance $2:5$ ($P_{\mathrm{Saturn}}/P_{\mathrm{Jupiter}} = 2.4825 \approx 5/2$). In the case of the resonance $2:5$, we keep in Eq. (5) only terms such that $(k_1, k_2) = (2k, -5k)$ for some integer $k \in \mathbb{Z}$.

Q2 - *Using d'Alembert rule, determine at which order d in eccentricity does resonance $2:5$ appear. More generally, at which order in eccentricity does resonance $p:q$ appear?*

In the planetary systems discovered to date, the MMR $2:3$ is very common. From now on, we will focus exclusively on this resonance.

Q3 - *Truncate H_P to first order in eccentricity ($d = 1$) and keep only terms such that $(k_1, k_2) = (2k, -3k)$ for some integer $k \in \mathbb{Z}$. Show that there exist coefficients $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$, depending on Λ_j, such that*

$$H_P = \mathcal{C}_0(\Lambda_j) + \mathcal{C}_1(\Lambda_j)\sqrt{P_1}\cos(2\lambda_1 - 3\lambda_2 - p_1) + \mathcal{C}_2(\Lambda_j)\sqrt{P_2}\cos(2\lambda_1 - 3\lambda_2 - p_2). \tag{8}$$

In order to work with variables adapted to this resonance, we will make a canonical linear transformation. Let us define $\boldsymbol{I} = {}^t(\Lambda_1, \Lambda_2, P_1, P_2)$ and $\boldsymbol{\theta} = {}^t(\lambda_1, \lambda_2, p_1, p_2)$ the current action-angle variables of the Hamiltonian. We define new action-angle variable $\boldsymbol{J} = {}^t(\Gamma, G, D_1, D_2)$ and $\boldsymbol{\varphi} = {}^t(\varphi_1, \varphi_2, \sigma_1, \sigma_2)$ with the transformation

$$\begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{\varphi} \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{M}_1 & 0 \\ \hline 0 & \mathcal{M}_2 \end{array}\right) \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{\theta} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{\theta} \end{pmatrix}. \tag{9}$$

This linear transformation is canonical if, and only if

$$\mathcal{M}\mathbb{J}\,{}^t\mathcal{M} = \mathbb{J}, \quad \text{where } \mathbb{J} = \left(\begin{array}{c|c} 0 & -\mathbb{I} \\ \hline \mathbb{I} & 0 \end{array}\right) \text{ and } \mathbb{I} \text{ is the identity matrix.} \tag{10}$$

Q4 - *Give the relation between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that the transformation is canonical.*

We choose

$$\mathcal{M}_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{pmatrix}. \tag{11}$$

Q5 - *Verify (with barely any calculations) that the Hamiltonian takes the form*

$$H(\boldsymbol{J},\sigma_1,\sigma_2)=H_K(\boldsymbol{J})+H_P(\boldsymbol{J},\sigma_1,\sigma_2). \tag{12}$$

In particular, the angles φ_1 and φ_2 do not appear in the Hamiltonian.

Q6 - *What can be said about the variables* Γ *and* G*? How many degrees of freedom are remaining in the Hamiltonian? What physical quantity is* G *equal to?*

2 Characteristic timescale of MMR 2:3

We define a transformation $(D_1,D_2;\sigma_1,\sigma_2)\rightarrow(R,S;r,s)$, called the Henrard-Sessin rotation, which, after some more manipulations, allows the Hamiltonian to take the form

$$H(R,S;r,s)=\alpha(S)R-\beta R^2+\gamma\sqrt{2R}\cos r, \tag{13}$$

where R and S are $\mathcal{O}(\Lambda_j e_j^2)$ eccentricity-related actions, whereas r and s are angles depending on σ_1 and σ_2. The quantities α, β and γ depend on the masses, on G and on Γ, whereas α also depends on S. They verify

$$\begin{aligned}\alpha&=\mathcal{O}(n_j),\\ \beta&=\mathcal{O}(n_j/\Lambda_j),\\ \gamma&=\mathcal{O}\left(\varepsilon n_j\Lambda_j^{1/2}\right),\end{aligned} \tag{14}$$

where $\varepsilon=(m_1+m_2)/m_0\ll 1$. More details about these manipulations can be found in this work[1], where the variables (R,S,r,s) and the quantities (α,β,γ) are properly defined.

Q7 - *How many degrees of freedom are remaining after the Henrard-Sessin transformation? What can be said about* S*? Justify that* α*,* β *and* γ *are constant parameters.*

The Hamiltonian of Eq. (13) is not a pendulum because of $\sqrt{2R}$ in front of $\cos r$. In its current form, this Hamiltonian is the so-called second fundamental model of resonance (SFM), first described by Henrard and Lemaitre, 1983. The first fundamental model of resonance refers to the simple pendulum. However, the SFM can be reduced to the pendulum by an additional approximation.

Q8 - *Show that Eq. (13) can be written*

$$H(R,r)=-\beta\left(R-R^\star\right)^2+\gamma\sqrt{2R}\cos r, \tag{15}$$

and give the expression of $R^\star$ *as a function of* α *and* β*. Remember that constants can freely be added/removed from a Hamiltonian without changing the equations of motions.*

The perturbation $\gamma\sqrt{2R}\cos r$ is ε-small with respect to the principal part $-\beta\left(R-R^\star\right)^2$.

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[1] https://jeremycouturier.com/img/PDF/SFM.pdf

We achieve the form of the pendulum by replacing the variable R by the constant $R^\star$ in the perturbation, and the pendulum model of MMR $2:3$ reads

$$H(R, r) = -\beta\,(R - R^\star)^2 + \gamma\sqrt{2R^\star}\cos r. \tag{16}$$

Q9 - *Write the equations of motions associated with this pendulum and find its two fixed points, that is to say, the two initial conditions from which the system does not move.*

In the neighborhood of a fixed point (R_0, r_0), we write $R = R_0 + dR$ and $r = r_0 + dr$.

Q10 - *Linearize, that is, expand to first order in dR and dr, the equations of motion in the neighborhood of each fixed point, and write them under the form*

$$\frac{d}{dt}\begin{pmatrix} dR \\ dr \end{pmatrix} = \mathcal{Q}\begin{pmatrix} dR \\ dr \end{pmatrix}. \tag{17}$$

The eigenvalues of a fixed point are, by definition, the eigenvalues of the constant matrix $\mathcal{Q}$. A fixed point is said to be elliptic if all of its eigenvalues are pure imaginary (proportional to ι), and hyperbolic otherwise.

Q11 - *Show that one fixed point is hyperbolic with eigenvalues $\pm\nu$, and the other elliptic with eigenvalues $\pm\iota\nu$, where $\nu = \sqrt{2\beta\gamma\sqrt{2R^\star}}$. What is the dimension and order of magnitude of ν? What physical quantity does it represent? Draw the phase space.*

3 Eccentricity excitation in the MMR 2:3

A consequence of being close to a first order MMR such as $2:3$ is the excitation of the eccentricities, which we will study now. For this, we go back to the SFM model provided by Eq. (13). While it seems that the SFM depends on three parameters (α, β, γ), the dependency is fictitious as it can be reduced to one parameter. We define a new angle $\sigma = r$, and we rescale R and the Hamiltonian with

$$\mathcal{H} = \frac{H}{A} \quad \text{and} \quad \Sigma = \frac{R}{B}. \tag{18}$$

Q12 - *Find A, B and δ such that the Hamiltonian reads*

$$\mathcal{H}(\Sigma; \sigma) = 3\delta\Sigma - \Sigma^2 + 2\sqrt{2\Sigma}\cos\sigma. \tag{19}$$

Although the variables $(R; r)$ are canonical, the rescaled variables (Σ, σ) are not.

Q13 - *Determine ω as a function of A and B such that, with the rescaled time $\tau = \omega t$, the equations of motions are under the canonical form*

$$\frac{d\Sigma}{d\tau} = -\frac{\partial\mathcal{H}}{\partial\sigma}, \quad \frac{d\sigma}{d\tau} = \frac{\partial\mathcal{H}}{\partial\Sigma}. \tag{20}$$

We will now work with the rectangular variables

$$X = \sqrt{2\Sigma}\cos\sigma, \quad Y = \sqrt{2\Sigma}\sin\sigma. \tag{21}$$

Q14 - *Show that the transformation* $(\Sigma, \sigma) \rightarrow (X, Y)$ *of Eq. (21), from polar to Cartesian variables, is canonical with momentum* X *and coordinate* Y*.*

Because of the rescaling, the variables X and Y are $\mathcal{O}\left(e_j\, \varepsilon^{-1/3}\right)$. The Hamiltonian now reads

$$\mathcal{H}(X;Y) = \frac{3}{2}\delta\left(X^2+Y^2\right) - \frac{1}{4}\left(X^2+Y^2\right)^2 + 2X. \tag{22}$$

Q15 - *Show that the fixed points are located at* $Y = 0$ *and* $X^3 - 3\delta X - 2 = 0$*.*

Because X is real, there is one fixed point when $\delta < 1$, and three fixed points when $\delta \geq 1$. I plot the phase space for different values of the parameter δ in Fig. 1. When $\delta < 1$, the unique fixed point is elliptic. When $\delta \geq 1$, there are two elliptic fixed points (black dots) and one hyperbolic fixed point. The red level line goes through the hyperbolic fixed point and is called the separatrix of the resonance. It intersects itself at the hyperbolic fixed point. A resonance is a region of the phase space bounded by a separatrix, where the resonant angle σ mostly librates.

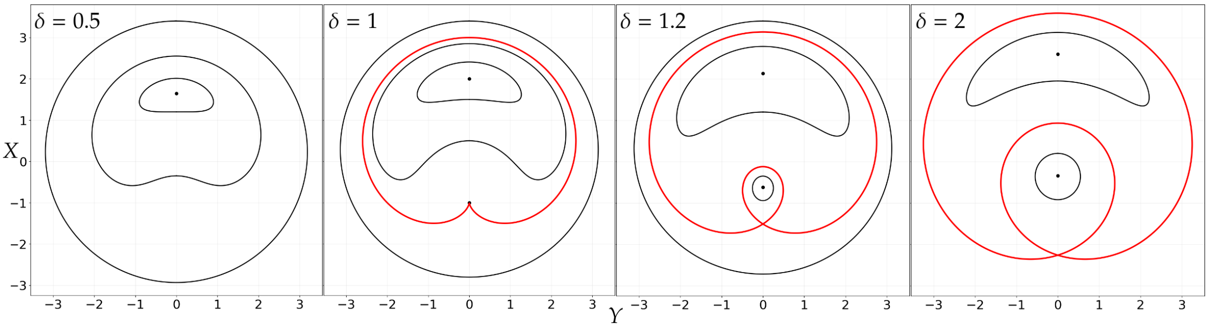

Figure 1 — Phase Space of the Hamiltonian (22) for some values of δ

Q16 - *Identify the resonant region in the phase space with* $\delta = 2$*.*

Because the parameter δ depends on the masses, different systems, or different analysis of a same system, cannot be plotted in Fig. 1, because the topology of the phase space depends on δ. Therefore, the phase space of the SFM is generally plotted on a figure when δ is on the x-axis, and the X-value of the level lines when they intersect $Y = 0$ are plotted on the y-axis.

In Fig. 2, I plot the two-planet system Kepler-1972 in the SFM, according to a posterior sample of Adrien Leleu. Such posterior sample is obtained by looking for masses and elliptic elements able to fit the observed transit light curves. The two elliptic fixed points are represented in Fig. 2 by the solid and dashed black lines. The hyperbolic fixed point is the dashed red lines. The two solid red lines are the intersections of the separatrix with $Y = 0$ in Fig. 1.

Q17 - *Identify the resonant region in Fig. 2.*

The solid grey curves in Fig. 2 are the level lines of the period ratio between the two planets. It can be seen that as the solid black fixed point gets closer to the commensurability

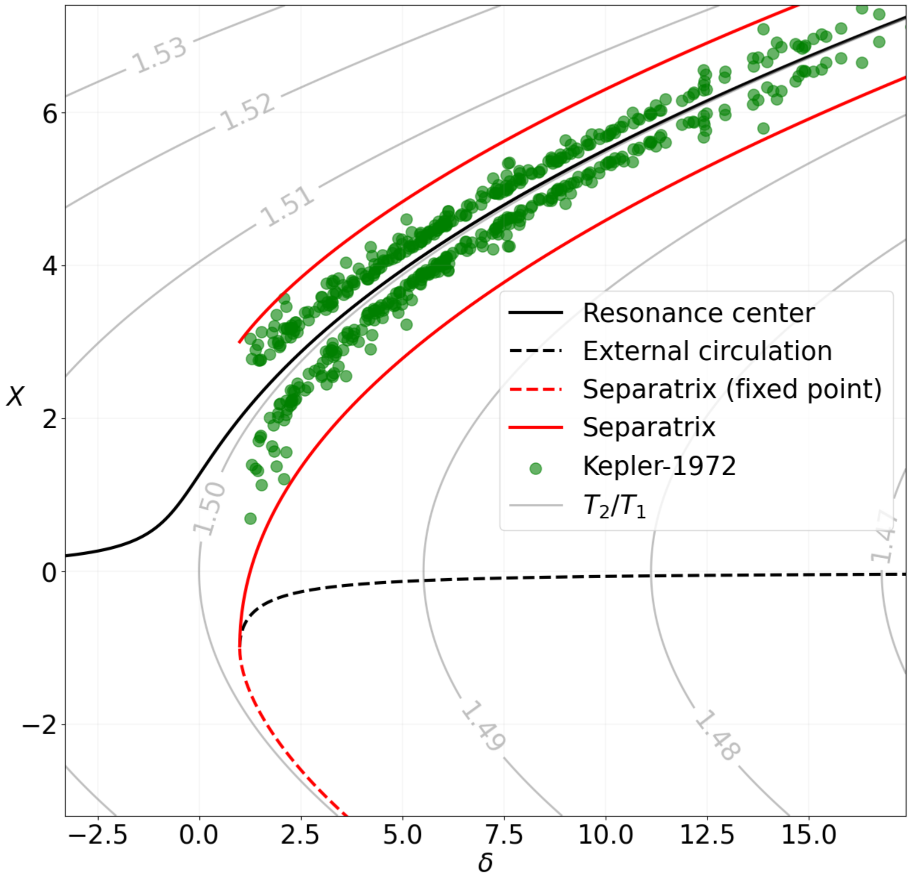

Figure 2 — Kepler-1972 in the SFM, according to a posterior analysis of Adrien Leleu based on transit light curves.

$T_2/T_1 = 3/2$, $X \propto e_j$ increases. The 2 : 3 MMR does indeed excite the eccentricities. Each point of Leleu's sample appears twice in Fig. 2, corresponding to the two intersections of the corresponding level line in Fig. 1 with $Y = 0$.

Q18 - *Is Kepler-1972 in resonance? What quantities are poorly constrained by the transit light curves and are responsible for the spread in Fig. 2?*

References

Henrard, J. and Lemaitre, A. (1983). A Second Fundamental Model for Resonance. *Celestial Mechanics*, 30, pp. 197–218.

Laskar, J. (2017). Andoyer Construction for Hill and Delaunay Variables. *Celestial Mechanics and Dynamical Astronomy*, 128, pp. 475–482.

Laskar, J. and Robutel, P. (1995). Stability of the Planetary Three-Body Problem. I. Expansion of the Planetary Hamiltonian. *Celestial Mechanics and Dynamical Astronomy*, 62, pp. 193–217.