The Eighth International Meeting on Celestial Mechanics

Co-orbital dynamics : *Tides and resonance chains*

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Co-orbital exoplanets are predicted by the formation models. Two scenarii of formation (Laughlin and Chambers, 2002)

- Accretion in situ at the Lagrangian point of a giant planet
  $\rightarrow$ maximum 0.6 Earth mass according to Beaugé et al. (2007).
  $\rightarrow$ 5 to 15 Earth mass according to Lyra et al. (2009).
- Capture at the L_{4-5} point of an already existing planet
  $\rightarrow$ a high diversity of mass ratio (Cresswell and Nelson, 2008)

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As of September 2, 2022, 5159 detected exoplanets.

$\rightarrow$ But no co-orbital exoplanet.

Can tidal dissipation be the cause ?

$\rightarrow$ We build a secular model of tidal dissipation in the $1:1$ planar MMR.

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Hamiltonian of the conservative problem

$$\mathcal{H} = \mathcal{H}_K(\Lambda_1, \Lambda_2) + H_P(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, x_1, x_2, \tilde{x}_1, \tilde{x}_2), \tag{1}$$

$$\mathcal{H}_K(\Lambda_1, \Lambda_2) = -\sum_{j=1}^{2} \frac{\beta_j^3 \mu_j^2}{2\Lambda_j^2}. \tag{2}$$

The semi-major axes stay close to a quantity denoted by $\bar{a}$.

$\rightarrow$ Expansion in the neighbourhood of

$$\tilde{}^{\star} = (\Lambda_1^\star, \Lambda_2^\star)\,, \quad \text{with} \quad \Lambda_j^\star = m_j\sqrt{\mu_0\,\bar{a}} \approx \Lambda_j. \tag{3}$$

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Averaging over the fast orbital frequency

- The dependency on λ_1 and λ_2 is lost.
- Dependency on $\xi = \lambda_1 - \lambda_2$ only.

Expansion of $\mathcal{H}_P$ over the eccentricities

$$\mathcal{H}_P = \sum_{n \geq 0} \mathcal{H}_{2n} \quad \text{where} \quad \mathcal{H}_{2n} = \sum_{|\boldsymbol{p}|=2n} \Psi_{\boldsymbol{p}}\left(\xi\right) X_1^{p_1} X_2^{p_2} \bar{X}_1^{\bar{p}_1} \bar{X}_2^{\bar{p}_2} \,. \tag{4}$$

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X_j is close to the eccentricity vector

$$X_j = e_j \exp(i\varpi_j) + \mathcal{O}(e_j^3). \qquad (5)$$

$\mathcal{H}_P$ is even in eccentricity $\rightarrow$ $X_j = 0$ is a stable manifold of the phase space.

What about the dynamics at zero eccentricity ?

We study the one-degree-of-freedom Hamiltonian $\mathcal{H}_K + \mathcal{H}_P(X_j = 0)$.

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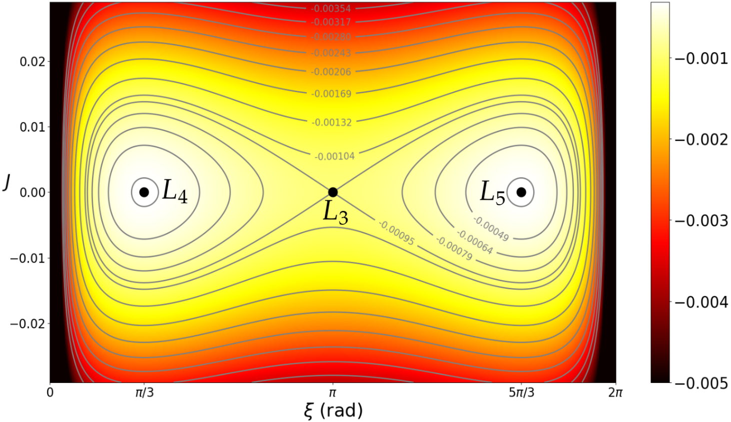
Phase space in the circular case

L_4 and L_5 are energy maximizers $\to$ L_4 and L_5 are tidally unstable.



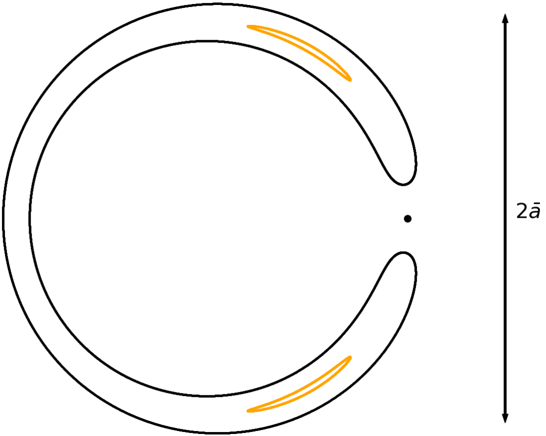


Figure: Position of m_1 with respect to m_2. Tadpole orange orbits : $\xi_0 = \pm 45°$. Horseshoe-shaped black orbit : $\xi_0 = 10°$.

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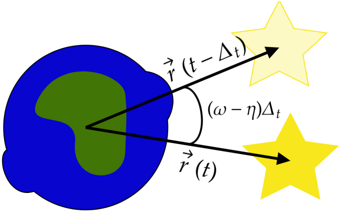
The linear model (Mignard, 1979).



Tidal potential (Kaula, 1964)

$$V(\boldsymbol{r}) = -\kappa_2^{(j)}\frac{\mathcal{G}m_i}{R_j}\left(\frac{R_j}{r}\right)^3\left(\frac{R_j}{r_i^\star}\right)^3 P_2(\cos S), \qquad \boldsymbol{r_i}^\star = \boldsymbol{r_i}(t-\Delta t_j).$$

Quality factor (MacDonald, 1964)

$$Q_j^{-1}(\eta) = \tan(\eta\Delta t_j(\eta)) \approx \eta\Delta t_j(\eta). \tag{6}$$

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Eqs. of motion are derived from a pseudo-Hamiltonian (dependency on t and on $t-\Delta t$)

We define the dissipation rate [1]

$$\text{dissipation rate of planet } j: \;\; q_j/Q_j = \kappa_2^{(j)}\frac{R_j^5}{\bar{a}^5}\eta\Delta t_j. \tag{7}$$

We obtain an equation of the form [1]

$$\dot{\mathfrak{X}} = F\left(\mathfrak{X}\right), \qquad \mathcal{X} = {}^t\left(\omega_1, \omega_2, J, J_2, \xi, X_1, X_2\right). \tag{8}$$

[1] *Couturier, Robutel & Correia, 2021*

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Linearization of the differential system

in the vicinity of its equilibria $\rightarrow$ Small perturbation of L_4 & L_5.

Linear system

$$\dot{\mathfrak{X}} = \left(\mathcal{Q}_0 + \mathcal{Q}_1\right)\mathfrak{X}. \tag{9}$$

$\mathcal{Q}_0$ conservative part, $\mathcal{Q}_1$ dissipative part.

$$\mathcal{Q}_{0\text{ or }1} = \begin{pmatrix} * & 0_{5,2} \\ 0_{2,5} & * \end{pmatrix} \tag{10}$$

$\rightarrow$ Even in the dissipative case, the dynamics on (X_1, X_2) is uncoupled from the rest in the vicinity of the equilibrium.

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In the absence of tides

Eigenvalues of $\mathcal{Q}_0$ (*Robutel & Pousse, 2013*)

$$\{0, 0, 0, i\nu, -i\nu, ig_1, ig_2\}, \tag{11}$$

$$\nu = \sqrt{\frac{27\,(m_1 + m_2)}{4m_0}}, \quad g_1 = \frac{27\,(m_1 + m_2)}{8m_0}, \quad g_2 = 0. \tag{12}$$

All the eigenvalues are pure imaginary → Quasi-periodic motion

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With tides

Eigenvalues of $\mathcal{Q}_0 + \mathcal{Q}_1$ (*Couturier, Robutel & Correia, 2021*)

$$\left\{\lambda_1, \lambda_2, 0, \reflectbox{λ}, \bar{\reflectbox{λ}}, \lambda_{\mathsf{AL}}, \lambda_{\mathsf{L}}\right\}, \tag{13}$$

with (real parts are boxed and $\iota = (m_1 + m_2)/m_0$)

$$\reflectbox{λ} = \boxed{\frac{9}{2}\iota^{-1}\left(\frac{m_1}{m_2}\frac{q_2}{Q_2} + \frac{m_2}{m_1}\frac{q_1}{Q_1}\right)} + i\nu\left[1 + 13\iota^{-1}\left(\frac{m_1}{m_2}q_2 + \frac{m_2}{m_1}q_1\right)\right],$$

$$\lambda_{\mathsf{AL}} = \boxed{-\frac{21}{2}\iota^{-1}\left(\frac{m_1}{m_2}\frac{q_2}{Q_2} + \frac{m_2}{m_1}\frac{q_1}{Q_1}\right)} + ig_1\left[1 + \frac{20}{9}\iota^{-2}\left(\frac{m_1}{m_2}q_2 + \frac{m_2}{m_1}q_1\right)\right],$$

$$\lambda_{\mathsf{L}} = \boxed{-\frac{21}{2}\iota^{-1}\left(\frac{q_1}{Q_1} + \frac{q_2}{Q_2}\right)} + \frac{15}{2}i\iota^{-1}(q_1 + q_2).$$

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Characteristic timescales of evolution of the co-orbital angle[1]

$$\tau_{\mathsf{lib}} = \frac{1}{9\pi}\frac{\varepsilon}{q_1/Q_1 + q_2/Q_2}\frac{x\,(1+y)}{1+yx^2}\,T \propto \frac{x\,(1+y)}{1+yx^2}\,\bar{a}^{6.5}. \tag{14}$$

with

$$x = \frac{m_1}{m_2} \text{ and } y = \frac{\text{dissipation rate}_2}{\text{dissipation rate}_1} = \frac{q_2\,Q_1}{q_1\,Q_2} \tag{15}$$

Time to horse-shoe shaped orbits

$$\tau_{\mathsf{hs}} = \ln\left(\frac{60^\circ}{\Delta\xi}\right)\tau_{\mathsf{lib}} \tag{16}$$

where $\Delta\xi$ is the initial angular distance to L_{4-5}

[1] *Couturier, Robutel & Correia, 2021*

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Co-orbital lifetime $\tau_{\textbf{dest}} \approx \tau_{\textbf{hs}}$

For two co-orbital Earth-like planets initially 0.1° away from L_4 :

$$\tau_{\text{hs}} = 3.612 \text{ Gyr } \left(\frac{\bar{a}}{0.04 \text{ AU}}\right)^{6.5} \left(\frac{m_0}{m_\odot}\right)^{-3/2}, \qquad (17)$$

Co-orbital pair	τ_{hs} (Gyr)	Co-orbital pair	τ_{hs} (Gyr)	Co-orbital pair	τ_{hs} (Gyr)
Earth & Earth	3.612	Moon & Mars	28.48	Mars & Saturn	5.000
Earth & Moon	43.81	Moon & Jupiter	50.63	Jupiter & Jupiter	0.7214
Earth & Mars	5.552	Moon & Io	4.374	Jupiter & Io	2.072
Earth & Jupiter	3.567	Moon & Saturn	43.13	Jupiter & Saturn	0.04741
Earth & Io	2.085	Mars & Mars	5.892	Io & Io	2.072
Earth & Saturn	1.803	Mars & Jupiter	5.878	Io & Saturn	2.054
Moon & Moon	50.76	Mars & Io	2.250	Saturn & Saturn	0.03704

Table: Destruction time using tidal parameters of (Lainey, 2016).

Top panel → Our model

Bottom panel → Full n-body problem

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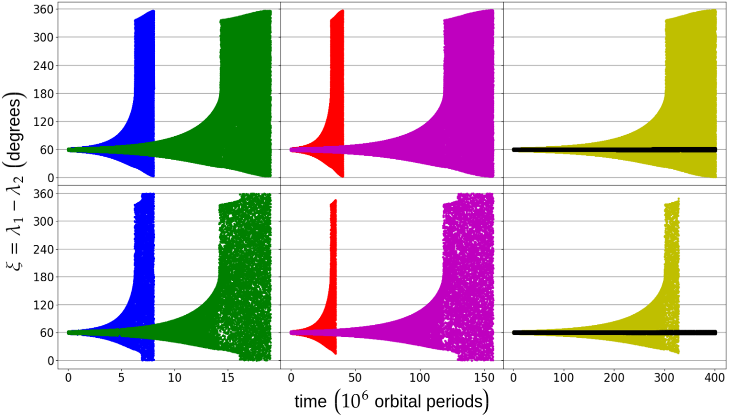
The libration amplitude of ξ unbounded



Less than 1% error on time to horseshoe-shaped orbits.

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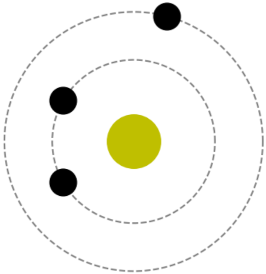
We introduce a third planet in the planetary system



First order MMR between the co-orbital pair and the third planet.

$$n_1^{(0)}/n_3^{(0)} = (p + 1)/p \tag{18}$$

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The resonance chain $p : p : p+1$

The Hamiltonian contains angles of the form

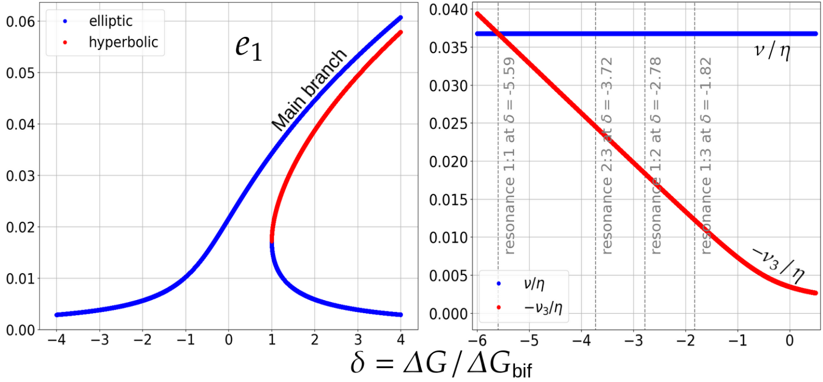
$$-p\lambda_1 + (p+1)\,\lambda_3 \quad \text{or} \quad -p\lambda_2 + (p+1)\,\lambda_3$$

and terms of the form

$$e_j \cos\left(-p\lambda_j + (p+1)\,\lambda_3 - \varpi_j\right) \text{ and } e_3 \cos\left(-p\lambda_j + (p+1)\,\lambda_3 - \varpi_3\right)$$

4 degrees of freedom associated with the angles

$$\begin{aligned}
\xi &= \lambda_1 - \lambda_2 \\
\sigma_1 &= -p\lambda_2 + (p+1)\,\lambda_3 - \varpi_1 \\
\sigma_2 &= -p\lambda_2 + (p+1)\,\lambda_3 - \varpi_2 \\
\sigma_3 &= -p\lambda_2 + (p+1)\,\lambda_3 - \varpi_3
\end{aligned}$$



The parameter δ is related to a_1/a_3, that is, to n_1/n_3

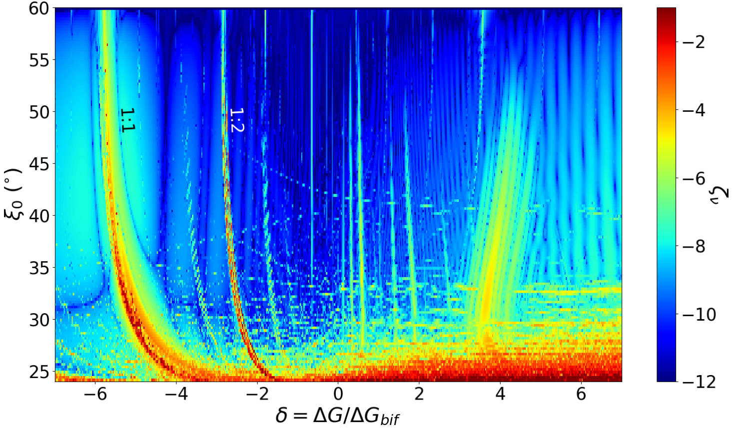
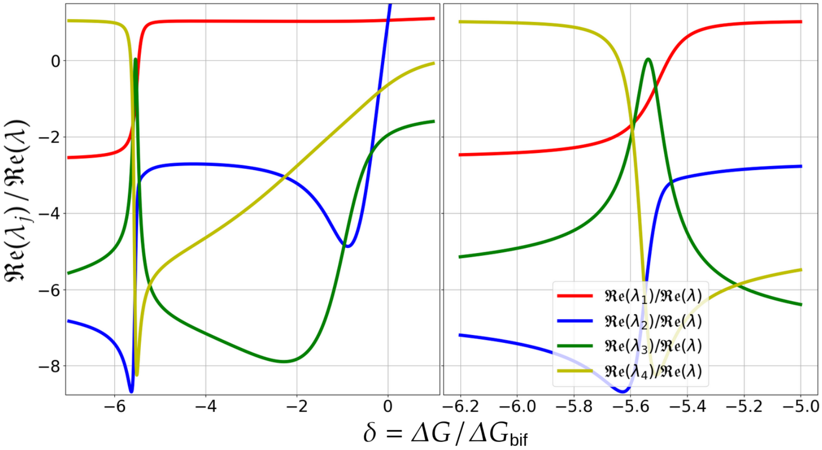
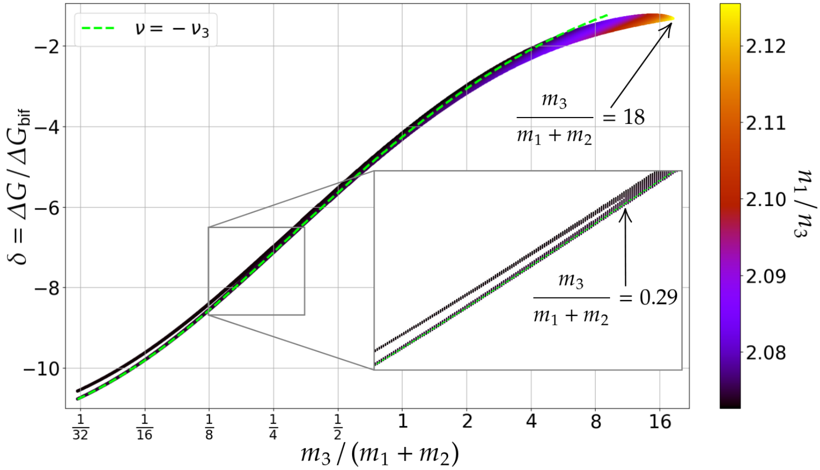


Figure: Diffusion of the fundamental frequencies (Laskar, 1990).

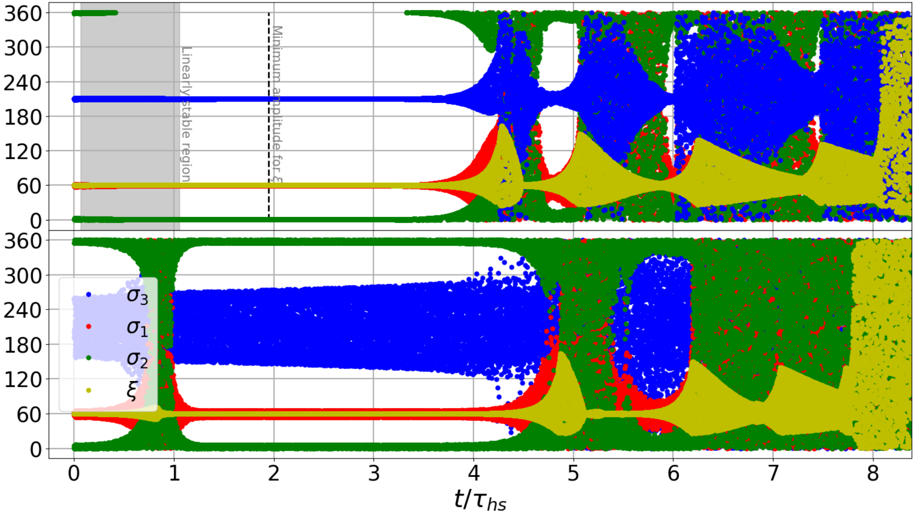
$\rightarrow 1:1$ **secular resonance between ν and ν_3.**

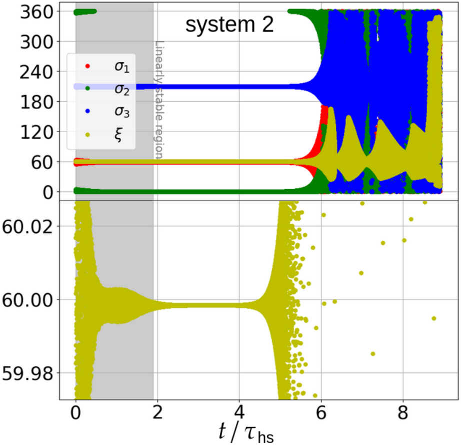


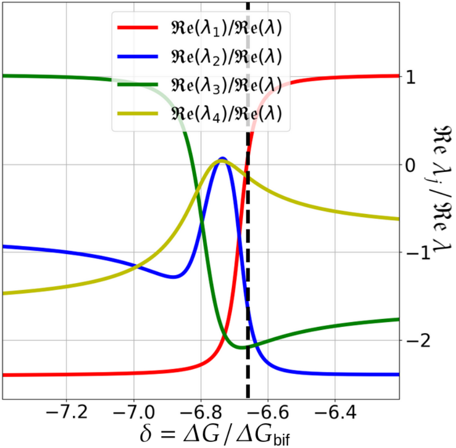
At the $1:1$ secular resonance between ν and ν_3 (that is, at $n_1/n_3 \approx 2.07$) $\rightarrow$ Linear stability

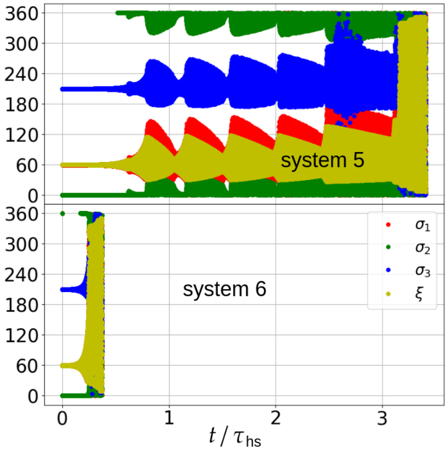


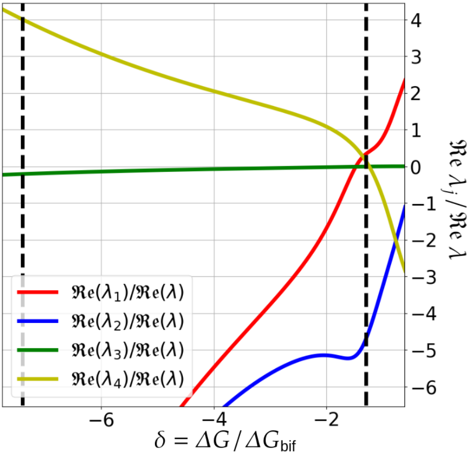
Position of the region of asymptotic stability

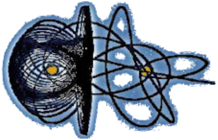

360
300
240
180
120
60
0
Linearly stable region
Minimum amplitude for ξ
σ3
σ1
σ2
ξ
0
1
2
3
4
5
6
7
8
t/τhs

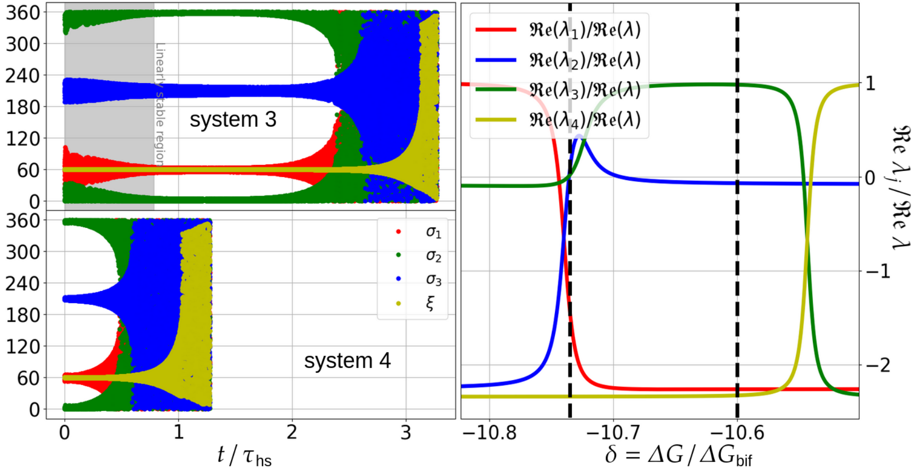

system 2
σ_1
σ_2
σ_3
ξ
Linearly stable region
360
300
240
180
120
60
0
60.02
60.00
59.98
0
2
4
6
8
$t\,/\,\tau_{\text{hs}}$


$\Re e(\lambda_1)/\Re e(\lambda)$
$\Re e(\lambda_2)/\Re e(\lambda)$
$\Re e(\lambda_3)/\Re e(\lambda)$
$\Re e(\lambda_4)/\Re e(\lambda)$
1
0
−1
−2
$\Re e\,\lambda_j\,/\,\Re e\,\lambda$
−7.2
−7.0
−6.8
−6.6
−6.4
$\delta = \Delta G\,/\,\Delta G_{\text{bif}}$


360
300
240
180
120
60
0
system 5
system 6
σ_1
σ_2
σ_3
ξ
$t\,/\,\tau_{\rm hs}$


$\Re(\lambda_1)/\Re(\lambda)$
$\Re(\lambda_2)/\Re(\lambda)$
$\Re(\lambda_3)/\Re(\lambda)$
$\Re(\lambda_4)/\Re(\lambda)$
$\delta = \Delta G\,/\,\Delta G_{\rm bif}$
$\Re\,\lambda_i\,/\,\Re\,\lambda$




system 3
Linearly stable region
system 4
σ_1
σ_2
σ_3
ξ
$t\,/\,\tau_{hs}$
$\Re e(\lambda_1)/\Re e(\lambda)$
$\Re e(\lambda_2)/\Re e(\lambda)$
$\Re e(\lambda_3)/\Re e(\lambda)$
$\Re e(\lambda_4)/\Re e(\lambda)$
$\Re e\,\lambda_i\,/\,\Re e\,\lambda$
$\delta = \Delta G\,/\,\Delta G_{bif}$

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Co-orbital exoplanets are always unstable under tides.

But they live long if :

- They orbit far away from the star $\rightarrow$ challenging detection
- The mass repartition is very inequal $\rightarrow$ challenging detection
- They are within the resonance chain $p : p : p + 1$

This work gives an argument for the absence of co-orbital exoplanet detection

Much more details in the associated papers :

- An analytical model for tidal evolution in co-orbital systems *10.1007/s10569-021-10032-w CM&DA*
- Dynamics of co-orbital exoplanets in a first order resonance chain with tidal dissipation *10.1051/0004-6361/202243261 A&A*

Thank you for your attention

Two different sets of equations

- Top plot $\to$ the secular equations derived in this work.
- Bottom plot $\to$ A direct n-body model in cartesian coordinates.

$$\frac{d^2\boldsymbol{r}_1}{dt^2} = -\frac{\mu_1}{r_1^3}\boldsymbol{r}_1 + \mathcal{G}m_2\left(\frac{\boldsymbol{r}_2-\boldsymbol{r}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^3} - \frac{\boldsymbol{r}_2}{r_2^3}\right) + \frac{\boldsymbol{f}_1}{\beta_1} + \frac{\boldsymbol{f}_2}{m_0} \ ,$$

$$\frac{d^2\boldsymbol{r}_2}{dt^2} = -\frac{\mu_2}{r_2^3}\boldsymbol{r}_2 + \mathcal{G}m_1\left(\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3} - \frac{\boldsymbol{r}_1}{r_1^3}\right) + \frac{\boldsymbol{f}_2}{\beta_2} + \frac{\boldsymbol{f}_1}{m_0} \ ,$$

$$\frac{d^2\theta_i}{dt^2} = -\frac{(\boldsymbol{r}_i\times\boldsymbol{f}_i)\cdot\boldsymbol{k}}{C_i} = -3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^3}{\alpha_i m_i r_i^8}\Delta t_i\left[\frac{d\theta_i}{dt}\,r_i^2 - \left(\boldsymbol{r}_i\times\frac{d\boldsymbol{r}_i}{dt}\right)\cdot\boldsymbol{k}\right] \ ,$$

$$\boldsymbol{f}_i = -3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^5}{r_i^8}\boldsymbol{r}_i$$

$$-\,3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^5}{r_i^{10}}\Delta t_i\left[2\left(\boldsymbol{r}_i\cdot\frac{d\boldsymbol{r}_i}{dt}\right)\boldsymbol{r}_i + r_i^2\left(\frac{d\theta_i}{dt}\,\boldsymbol{r}_i\times\boldsymbol{k} + \frac{d\boldsymbol{r}_i}{dt}\right)\right] \ .$$

The secular equations

$$\dot{\vartheta}_j = -3\alpha_j^{-1}\frac{m_0}{m_j}\varrho_j^{-2}\frac{q_j}{Q_j}\mathcal{R}_j^{-12}\left\{\vartheta_j + 3\left(1-\mathcal{R}_j\right) + h_2^j\mathcal{R}_j^{-1}X_j\bar{X}_j\right.$$

$$\left. + h_4^j\mathcal{R}_j^{-2}X_j^2\bar{X}_j^2\right\},$$

$$\dot{J} = -\frac{\partial\left(\mathcal{H}_0+\mathcal{H}_2+\mathcal{H}_4\right)}{\partial\xi} + (1-\delta)\,\dot{J}_2^1 - \delta\dot{J}_2^2,$$

$$\dot{J}_2 = \dot{J}_2^1 + \dot{J}_2^2,$$

$$\dot{\xi} = \frac{\partial\mathcal{H}_K}{\partial J} + 6\,q_1\frac{m_0}{m_1}\mathcal{R}_1^{-13}\,V_2\left(\mathcal{R}_1^{-1}X_1\bar{X}_1\right) - 6\,q_2\frac{m_0}{m_2}\mathcal{R}_2^{-13}\,V_2\left(\mathcal{R}_2^{-1}X_2\bar{X}_2\right),$$

$$\dot{X}_j = -2i\frac{m}{m_j}\frac{\partial\left(\mathcal{H}_2+\mathcal{H}_4\right)}{\partial\bar{X}_j}$$

$$-\,3\frac{q_j}{Q_j}\frac{m_0}{m_j}\mathcal{R}_j^{-13}X_j\left\{p_2^j - \frac{5}{2}iQ_j + \frac{X_j\bar{X}_j}{\mathcal{R}_j}\left(p_4^j - \frac{65}{4}iQ_j\right)\right\},$$